Is critical 2D percolation universal?

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Abstract. The aim of these notes is to explore possible ways of extending Smirnov's proof of Cardy's formula for critical site-percolation on the triangular lattice to other cases (such as bond-percolation on the square lattice); the main question we address is that of the choice of the lattice embedding into the plane which gives rise to conformal invariance in the scaling limit. Even though we were not able to produce a complete proof, we believe that the ideas presented here go in the right direction.

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Introduction

It is a strongly supported conjectured that many discrete models of random media, such as *e.g.* percolation and the Ising model, when taken in dimension 2 at their critical point, exhibit conformal invariance in the scaling limit. Indeed, the *universality* principle implies that the asymptotic behavior of a critical system after rescaling should not depend on the specific details of the underlying lattice, and in particular it should be invariant under rotations (at least under suitable symmetry conditions on the underlying lattice). Since by construction a scaling limit is also invariant under rescaling, it is natural to expect conformal invariance, as the local behavior of a conformal map is the composition of a rotation and a rescaling.

On the other hand, conformally invariant *continuous* models have been thoroughly studied by physicists, using tools such as conformal field theories. In 2000, Oded Schramm ([14]) introduced a one-parameter family of continuous bidimensional random processes which he called SLE processes, as the only possible scaling limits in this situation, under the assumption of conformal invariance; connections between SLE and CFT are now quite well understood (see *e.g.* [7, 1]).

However, actual convergence of discrete models to SLE in the scaling limit is known for only a few models. The case on which we focus in this paper is that

of percolation. The topic of conformal invariance for percolation has a long history — see [13] and references therein for an in-depth discussion of it.

In the case of site-percolation on the triangular lattice, it is a celebrated result of Smirnov ([16]) that indeed the limit exists and is conformally invariant. While the proof is quite simple and extremely elegant (see section 3 below and references therein), it is very specific to that particular lattice, to the point of being almost magical; it is a very natural question to ask how it can be generalized to other cases, and in particular to bond-percolation on the square lattice. Universality and conformal invariance have indeed been tested numerically for percolation in various geometries (see e.g. [12]), and conformal invariance (assuming the existence of the limit) is known in the case of Voronoi percolation (see [4]).

In fact, it seems that the question of convergence itself has hardly been addressed by physicists, at least in the CFT community — a continuous, conformally invariant object is usually the starting point of their work rather than its outcome. Techniques such as the renormalization group do give reason to expect the existence of a scaling limit and of critical exponents, but they seem to not give much insight into the emergence of rotational invariance.

This is not surprising in itself, for a very trivial reason: Take any discrete model for which you know that there is a conformally invariant scaling limit, say a simple random walk on $\mathbb{Z}^2$, and deform the underlying lattice, in a linear way, so as to change the aspect ratio of its faces. Then the scaling limit still exists (it is the image of the previous one by the same transformation); but obviously it is not rotationally invariant. Since all the rescaling techniques apply exactly the same way before and after deformation, they cannot be sufficient to derive rotational invariance. A trace of this appears in the most general statement of the universality hypothesis (see *e.g.* [13, section 2.4]): To paraphrase it, given any two periodic planar graphs, the scaling limits of critical percolation on them are conjugated by *some* linear map g.

The main question we address in these notes is the following: Given a discrete model on a doubly periodic planar graph, how to embed this graph into the plane so as to make the scaling limit isotropic? If the graph has additional symmetry (as for instance in the case of the square or triangular lattices), the embedding has to preserve this symmetry; so a restating of the same question in the terms of the universality hypothesis would be, absent any additional symmetry for one of the two graphs involved, can one determine the map g?

The most surprising thing (to me at least) about the question, besides the fact that it appears to actually be orthogonal to the interests of physicists in that domain, is that its answer turns out to depend on the model considered. In other words, there is no absolute notion of a "conformal embedding" of a general graph. In the case of the simple random walk, the answer is quite easy to obtain, though it does not seem to have appeared in the literature in the form we present it here; in the case of percolation, I could find no reference whatsoever, the closest being

the discussion and numerical study of *striated models* in [13] where, instead of looking at a different graph, the parameter p in the model is chosen to depend on the site in $\mathbb{Z}^2$ in a periodic fashion — which admittedly is a very related question.

The paper is roughly divided into two parts. In the first one, comprised of the firs two sections, we introduce some notation and the general framework of the approach, and we treat the case of the simple random walk. This is enough to prove that the correct embedding is not the same for it as for percolation; we then argue that circle packings might give a way to answer the question in the latter case. In the second part, which is of a more speculative nature, we investigate Smirnov's proof in some detail, and rephrase it in such a way that its general strategy can be applied to general triangulations. We then describe the two main steps of a strategy that could lead to its generalization, though we were able to perform none of the two.

1. Notation and setup

1.1. The graph

We first define the class of triangulations of the plane we are interested in. Let T be a 3-regular finite graph of genus 1 (*i.e.*, a graph that is embeddable in the torus $\mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$ but not in the plane, and having only vertices of degree 3). For ease of notation, we assume that T is equipped with a fixed embedding in $\mathbb{T}^2$, which we also denote by T. The dual T^* of T (which we also assume to be embedded in the torus once and for all) is then a triangulation of $\mathbb{T}^2$.

Let $\hat{T}$ (resp. $\hat{T}^*$) be the universal cover of T (resp. T^*): Then $\hat{T}$ and $\hat{T}^*$ are mutually dual, infinite, locally finite planar graphs, on which $\mathbb{Z}^2$ acts by translation. We are interested in natural ways of embedding $\hat{T}$ into the complex plane $\mathbb{C}$. Let T_i (the meaning of the notation will become clear in a minute) be the embedding obtained by pulling T back using the canonical projection from $\mathbb{R}^2$ to $\mathbb{T}^2$ — we will call T_i the *square embedding* of T.

For every $\alpha \in \mathbb{C} \setminus \mathbb{R}$, let $\varphi_\alpha : \mathbb{C} \to \mathbb{C}$ be the $\mathbb{R}$-linear map defined by $\varphi_\alpha(x+iy) = x + \alpha y$ (*i.e.*, it sends 1 to itself and i to α) and let T_α be the image of T_i by φ_α. For lack of a better term, we will call T_α the *embedding of modulus* α of T in the complex plane.

Notice that the notation T_α depends on the *a priori* choice of the embedding of T in the flat torus; but, up to rotation and scaling, the set of proper embeddings of $\hat{T}$ obtained starting from two different embeddings of T is the same, so no generality is lost (as far as our purpose in these notes is concerned).

One very useful restriction on embeddings is the following:

Definition 1. We say that an embedding T_α of $\hat{T}$ in the complex plane is *balanced* if each of its vertices is the barycenter (with equal weights) of its neighbors; or, equivalently, if the simple random walk on it is a martingale.

Proposition 2. *Let T be a 3-regular graph of genus 1: Then, for every $\alpha \in \mathbb{H}$, there is a balanced embedding of $\hat{T}$ in the complex plane with modulus α. Moreover, this embedding is unique up to translations of the plane.*

Proof. We only give a sketch of the proof, because expanding it to a full proof is both straightforward and tedious. The main remark is that any periodic embedding which minimizes the sum S_2, over a period, of the squared lengths of its edges is balanced: Indeed, the gradient, with respect to the position of a given vertex, of S_2 is exactly the difference between this point and the barycenter of its neighbors. (This would be true in any Euclidean space.) It is easy to use a compactness argument to prove the existence of such a minimizer.

To prove uniqueness up to translation is a little trickier, but since it is not necessary for the rest of this paper, we allow ourselves to give an even sketchier argument. First, one can get rid of translations by assuming that a fixed vertex of $\hat{T}$ is put at the origin by the embedding; the set of all possible embeddings of modulus α is then parameterized by $2(|V(T)|-1)$ real-valued parameters, which are the coordinates of the locations of the other vertices in one period of $\hat{T}$. In terms of these variables, S_2 is polynomial of degree 2. It is bounded below by the squared length of the longest edge in the embedding, which itself is bounded below, up to a constant depending only on the combinatorics of the graph, by the square of the largest of the $2(|V(T)|-1)$ parameters; so it goes to infinity uniformly at infinity. This implies that its Hessian (which is constant) is positive definite, so S_2 is strictly convex as a function of those variables. This immediately implies the uniqueness of the minimizer. $\square$

An essential point is that, even though our proof uses Euclidean geometry, the fact that the embedding is balanced is a linear condition. In particular, if the embedding T_i is balanced, then so are all the other T_α. The corresponding *a priori* embedding of T itself into the flat torus $\mathbb{T}^2$ (which is also unique up to translations) will be freely referred to as *the balanced embedding of T into the torus.*

1.2. The probabilistic model

We will be interested in critical site-percolation on the triangulation T_α^*; more specifically, the question we are interested is the following. Let Ω be a simply connected, smooth domain in the complex plane, and let A, B, C and D be four points on its boundary, in that order. For every $\delta > 0$, let Ω_δ be the largest connected component (in terms of graph connectivity) of the intersection of Ω with δT_α, and Ω_δ^* be its dual graph. Ω_δ should be seen as a discretization of Ω at scale δ. Let A_δ, B_δ, C_δ and D_δ be the vertices of Ω_δ that are closest to A, B, C and D respectively.

The model we are most interested in is critical site-percolation on Ω_δ^*; however, most of the following considerations remain valid for other lattice models. Let $C_\delta(\Omega, A, B, C, D)$ be the event that there is an open crossing in Ω_δ^*, between the intervals $A_\delta B_\delta$ and $C_\delta D_\delta$ of its boundary. Under some symmetry conditions on T, Russo-Seymour-Welsh theory ensures that at criticality, the probability of

$C_\delta(\Omega, A, B, C, D)$ is bounded away from both 0 and 1 as δ goes to 0. Its limit was conjectured by Cardy (see [6]) using non-rigorous arguments from conformal field theory; actual convergence was proved, in the case of the triangular lattice (embedded in such a way that its faces are equilateral triangles), by Smirnov (see [15, 3]). We defer the statement of the convergence to a later time. The following definition has become standard:

Definition 3. Assume that, for every choice of (Ω, A, B, C, D), the probability of the event $C_\delta(\Omega, A, B, C, D)$ has a limit $f_\alpha(\Omega, A, B, C, D)$ as $\delta \to 0$ — we will refer to this by saying that the model *has a scaling limit*. We say that the model is *conformally invariant in the scaling limit* if, for every conformal map Φ from Ω to $\Phi(\Omega)$, one has

$$f_\alpha(\Omega, A, B, C, D) = f_\alpha(\Phi(\Omega), \Phi(A), \Phi(B), \Phi(C), \Phi(D)).$$

This is equivalent to saying that $f_\alpha(\Omega, A, B, C, D)$ only depends on the modulus of the conformal rectangle (Ω, A, B, C, D).

(Notice that the extension of Φ to the boundary of Ω, which is necessary for the above definition to make sense, is ensured as soon as Ω is assumed to be regular enough.)

2. Periodic embeddings

2.1. Uniqueness of the modulus

Given T, it is natural to ask whether it is possible to choose a value for α which provides conformal invariance in the scaling limit. There are two possible strategies: Either give an explicit value for which “a miracle occurs” (in physical terms, for which the model is *integrable* — this is what Smirnov did in the case of the triangular lattice), or obtain its existence in a non-constructive way — which is what we are trying to do here.

A reassuring fact is that, whenever such an α exists, it is essentially unique:

Proposition 4. *For every graph T, there are either zero or two values of α such that critical site-percolation on T_α^* is conformally invariant in the scaling limit. In the latter case, the two values are complex conjugates of each other.*

Proof. The key remark is the following: Let β be a non-real complex number. Since the event C_δ is defined using purely combinatorial features, one can push the whole picture forward through φ_β without changing its probability. Let $\alpha' = \varphi_\beta(\alpha)$: φ_β then transforms Ω into $\varphi_\beta(\Omega)$ and the lattice T_α into $T_{\alpha'}$. So, assuming convergence on both sides, one always has

$$f_\alpha(\Omega, A, B, C, D) = f_{\alpha'}(\varphi_\beta(\Omega), \varphi_\beta(A), \varphi_\beta(B), \varphi_\beta(C), \varphi_\beta(D)).$$

In the case $\beta = -i$, φ_β is simply the map $z \mapsto \bar{z}$. In that case, the modulus of the conformal rectangle $(\varphi_{-i}(\Omega), \bar{D}, \bar{C}, \bar{B}, \bar{A})$ is the same as that of (Ω, A, B, C, D), and clearly the event C_δ is invariant when the order of the corners is reversed. So,

conformal invariance for T_α and the previous remark implies that $f_{\bar\alpha}(\Omega, A, B, C, D)$ still only depends on the modulus of the conformal rectangle — in other words, if critical percolation T_α is conformally invariant in the scaling limit, that is also the case on $T_{\bar\alpha}$.

Now assume conformal invariance in the scaling limit for two choices of the modulus in the upper-half plane; these moduli can always be written as α and $\alpha' = \varphi_\beta(\alpha)$ for an appropriate choice of $\beta \in \mathbb{H} \setminus \{i\}$. Still using the above remark, all that is needed to arrive to a contradiction is to show that f_α does actually depend on the modulus of the rectangle (*i.e.*, that it is not constant), and that there exist two conformal rectangles with the same modulus and whose images by φ_β have different moduli.

For the former point, it is enough to prove that for every choice of $\rho, \rho' > 0$, the probability of crossing the rectangle $[0,\rho] \times [0,1]$ horizontally is strictly larger than that of crossing $[0, \rho+\rho'] \times [0,1]$. This is obvious by Russo-Seymour-Welsh: The event that there is a vertical dual crossing in $\delta T^*_\alpha \cap [\rho+\delta, \rho+\rho'] \times [0,1]$ is independent of $C_\delta([0,\rho]\times[0,1], \rho, \rho+i, i, 0)$ and its probability is bounded below, uniformly in $\delta < \rho'/10$, by some positive ε depending only on ρ and ρ'. Hence, still assuming that the limits all exist as $\delta \to 0$,

$$f_\alpha([0,\rho+\rho']\times[0,1], \rho+\rho', \rho+\rho'+i, i, 0) \leqslant (1-\varepsilon) f_\alpha([0,\rho]\times[0,1], \rho, \rho+i, i, 0).$$

For the latter point, assume that φ_β preserves the equality of moduli of conformal rectangles. Let $Q = [0,1]^2$ be the unit square. By symmetry, the conformal rectangles $(Q, 0, 1, 1+i, i)$ and $(Q, 1, 1+i, i, 0)$ have the same modulus; on the other hand $\varphi_\beta(Q)$ is a parallelogram, and by our hypothesis on φ_β it has the same modulus in both directions. This easily implies that it is in fact a rhombus. If now Q' is the square with vertices $1/2$, $1+i/2$, $1/2+i$, $i/2$, $\varphi_\beta(Q')$ is both a rhombus (by the same argument) and a rectangle (because its vertices are the midpoints of the edges of $\varphi_\beta(Q)$ which is a rhombus). Hence $\varphi_\beta(Q')$ is a square, and so is $\varphi_\beta(Q)$, and in particular $\beta = \varphi_\beta(i) = i$, which is in contradiction with our hypothesis. □

When such a pair of moduli exists, we will denote by α_T^{perc} the one with positive imaginary part. The same reasoning can be done for various models, and in each case where the scaling limit exists and is non-trivial, there will be a pair of moduli making it conformally invariant; we will distinguish them from each other by using the name of the model as a superscript (so that for instance α_T^{RW} makes the simple random walk conformally invariant in the scaling limit — cf. below).

When an argument does not depend on the specific model (as is the case in the next subsection), we will use the generic notation α_T as a placeholder.

2.2. Obtaining α_T by symmetry arguments

It should be noted that, because the value of α_T (when it exists) is uniquely defined by the combinatorics of T, there are cases where additional symmetry specifies its value uniquely. Indeed, assume Ψ is a graph isomorphism of $\hat{T}$ which is neither a translation nor a central symmetry; for every α, it induces a topological isomorphism of T_α. Assume without loss of generality that the origin of the plane

is chosen to be one of the vertices of T_α; let $z_0 = \Psi(0)$, $z_1 = \Psi(1)$ and $z_\alpha = \Psi(\alpha)$ (notice that both 1 and α are also vertices of T_α).

Assume $\alpha = \alpha_T^{\mathrm{perc}}$. Because Ψ is an isomorphism, it preserves site-percolation; so, in particular, critical site-percolation on $\Psi(T_\alpha)$ is conformally invariant in the scaling limit. By Proposition 4, this implies that

$$\frac{z_\alpha - z_0}{z_1 - z_0} = \frac{\Psi(\alpha) - \Psi(0)}{\Psi(1) - \Psi(0)} \in \{\alpha, \bar{\alpha}\}. \tag{2.1}$$

This condition is then enough to obtain the value of α_T. There are two natural examples of that (illustrated in Figures 1 and 2), which we now describe.

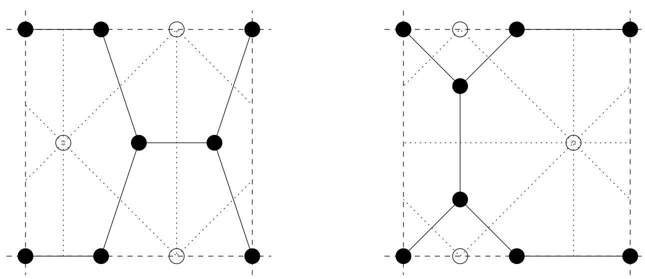

FIGURE 1. The graphs T_h (left) and T_s (right), embedded into $\mathbb{T}^2$ in a balanced way with a vertex at the origin; empty circles and dotted lines represent the dual graphs. Both are represented using their square embedding, so the triangles in T_h are not equilateral.

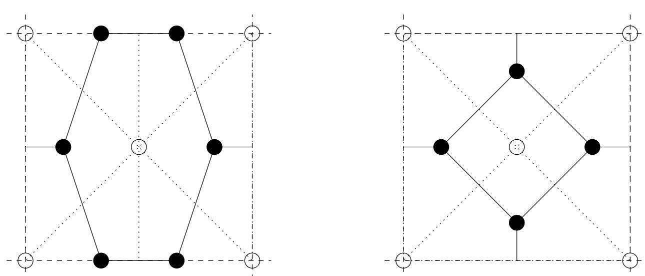

FIGURE 2. The same graphs as in Figure 1, with the origin on a vertex of the dual.

- Let T_h be one period of the honeycomb lattice, embedded into $\mathbb{T}^2$ in such a way that every vertex is the barycenter of its neighbors (we will call such an embedding *balanced*); since we take $\mathbb{T}^2$ to be a square, the coordinates of the vertices of T_h are $(0,0)$, $(1/3,0)$, $(1/2,1/2)$ and $(5/6,1/2)$. There is an

isomorphism Ψ_h of order 3 of $\hat{T}_h$, corresponding to rotation around $(0,0)$; on T_α, it sends 0 to $z_0 = 0$, 1 to $z_1 = (3\alpha - 1)/2$ and α to $z_\alpha = -(1+\alpha)/2$. Since Ψ_h preserves orientation, Equation (2.1) leads to

$$\frac{z_\alpha - z_0}{z_1 - z_0} = \frac{1+\alpha}{1-3\alpha} = \alpha \qquad \Longrightarrow \qquad \alpha = \pm i\frac{\sqrt{3}}{3},$$

in other words $\alpha_{T_h} = i\sqrt{3}/3$. Not surprisingly, this corresponds to embedding the faces of $\hat{T}_h^*$ as equilateral triangles, and those of $\hat{T}_h$ as regular hexagons.

- Let T_s be chosen in such a way that $\hat{T}_s^*$ has the topology of the centered square lattice; if again the embedding is balanced, the coordinates of the vertices of T_s are $(0,0)$, $(1/2,0)$, $(1/4,1/4)$ and $(1/4,3/4)$. There is an isomorphism Ψ_s of order 4 of $\hat{T}_s$, corresponding to a rotation rotation around the vertex $(1/4,0)$ of T_s^*. In that case

$$\frac{z_\alpha - z_0}{z_1 - z_0} = \frac{(-3/4 - \alpha/4) - (1/4 - \alpha/4)}{(1/4 + 3\alpha/4) - (1/4 - \alpha/4)} = \frac{-1}{\alpha} = \alpha \qquad \Longrightarrow \qquad \alpha = \pm i,$$

so $\alpha_{T_s} = i$. Again not surprisingly, this corresponds to the usual embedding of the square lattice using — well, squares.

Of course, identifying α_T in those cases is a long way from a proof of conformal invariance; but it would seem that understanding, in the general case, what α_T^{perc} is would be a significant progress in our understanding of the process.

2.3. Embedding using random walks

As an aside, in this subsection and the next we describe two natural ways of embedding a doubly periodic graph into the complex plane, which both have something to do with conformal invariance.

Let T be a finite 3-regular graph of genus 1, embedded in $\mathbb{T}^2$ in a balanced way, and let $(X_n)_{n\geqslant 0}$ be a simple random walk on it. For simplicity, assume that (X_n) is irreducible as a Markov chain. (Both 3-regularity and irreducibility are completely unnecessary as far as the results presented here are concerned, and the same reasoning would work in the general case, but notation would be a little tedious.) Since T is finite, (X_n) converges in distribution to the unique invariant measure, which, because T is 3-regular, is the uniform measure on $V(T)$; moreover the convergence is exponentially fast.

Now pick $\alpha \in \mathbb{H}$, and lift (X_n) to a simple random walk (Z_n) on T_α. By the balance condition on the embedding, it is easy to check that (Z_n) is a martingale; exponential decay of correlations between its increments is enough to obtain a central limit theorem (cf. for instance [9] and references therein). To write the covariance matrix in a convenient form, we need some notation. For each (oriented) edge e of T, choose $z_1(e)$ and $z_2(e)$ in T_α in such a way that they are neighbors and the edge $(z_1(e), z_2(e))$ is a pre-image of e by the natural projection from T_α

to T; let $e_\alpha := z_2(e) - z_1(e)$ — obviously it does not depend on the choice of $z_1(e)$ and $z_2(e)$. Define

$$\Sigma_\alpha^{xx}(T) := \frac{1}{|E(T)|} \sum_{e\in E(T)} (\Re e_\alpha)^2,$$

$$\Sigma_\alpha^{yy}(T) := \frac{1}{|E(T)|} \sum_{e\in E(T)} (\Im e_\alpha)^2,$$

$$\Sigma_\alpha^{xy}(T) := \frac{1}{|E(T)|} \sum_{e\in E(T)} (\Re e_\alpha)(\Im e_\alpha).$$

It is not difficult to compute the covariance matrix of the scaling limit of the walk:

Proposition 5. *As n goes to infinity, $n^{-1/2}Z_n$ converges in distribution to a Gaussian variable with covariance matrix*

$$\Sigma_\alpha(T) := \begin{bmatrix} \Sigma_\alpha^{xx}(T) & \Sigma_\alpha^{xy}(T) \\ \Sigma_\alpha^{xy}(T) & \Sigma_\alpha^{yy}(T) \end{bmatrix}.$$

Proof. The walk is centered by definition; the existence of a Gaussian limit is a direct consequence of the exponential decay of step correlations. All that remains to be done is to compute the covariance matrix. We focus on the first matrix entry, the others being similar. We have

$$\mathrm{Var}(n^{-1/2}\Re Z_n) = E\left[\frac{1}{n}\left(\sum_{k=0}^{n-1} \Re(Z_{k+1}-Z_k)\right)^2\right] = \frac{1}{n}\sum_{k=0}^{n-1} E\left[(\Re(Z_{k+1}-Z_k))^2\right]$$

(the other terms disappear by the martingale property). We know that $Z_{k+1}-Z_k$ converges in distribution, because the walk on T converges in distribution; its limit is the distribution of e_α where e is an edge of T chosen uniformly. By Cesàro's Lemma, the expression above then converges to Σ_α^{xx}; the computation of the other entries in $\Sigma_\alpha(T)$ it exactly similar. □

Even though the previous definition of conformal invariance in the scaling limit does not apply directly in this case, its natural counterpart is to ask for the scaling limit of the walk to be rotationally invariant (*i.e.*, to be standard two-dimensional Brownian motion); this is equivalent to saying that the covariance matrix $\Sigma_\alpha(T)$ is scalar, and since its entries are real, yet another equivalent formulation is

$$[\Sigma_\alpha^{xx}(T) - \Sigma_\alpha^{yy}(T)] + i\,[\Sigma_\alpha^{xy}(T)] = 0 \qquad \Longleftrightarrow \qquad \sum_{e\in E(T)} (e_\alpha)^2 = 0.$$

The last equation is a second-degree equation in α with real-valued coefficients. If $\alpha \in \mathbb{R}$, all the terms are non-negative and at least one is positive, so the equation has no solution in $\mathbb{R}$; letting α go to $+\infty$ along the real line leads to $|E(T)|$ positive terms, at least one of which is of order α^2, so the coefficient in α^2 in the equation is not zero. Hence the equation has exactly two solutions

which are complex conjugate of each other — the situation is very similar to the one in Proposition 4. For further reference, we let α_T^{RW} be the one with positive imaginary part. One advantage of this choice (besides the fact that it exists for every doubly periodic graph) is that the value of α_T^{RW} is very easy to compute.

Remark 6. In the more general case of a doubly periodic graph but without the assumptions of 3-regularity and irreducibility (but still assuming that the embedding is balanced), the condition $\sum e_\alpha^2$ is still necessary and sufficient for the walk to be isotropic in the scaling limit — and the proof is essentially the same, so we do not delve into more detail.

Remark 7. Of course, in the cases where T has some additional symmetry, α_T^{RW} is the same as that obtained in the previous subsection using symmetry ...

Remark 8. One can also look at a simple random walk on the dual graph T_α^*, and ask for which values of α this dual walk is isotropic in the scaling limit. As it turns out, the modulus one obtains this way is the same as on the initial graph, in other words

$$\alpha_T^{\mathrm{RW}} = \alpha_{T^*}^{\mathrm{RW}}.$$

This is a very weak version of universality, and unfortunately there doesn't seem to be a purely discrete proof of it — say, using a coupling of the two walks.

There is another natural way to obtain the same condition. We are planning on studying convergence of discrete objects to conformally invariant limits, so it is a good idea to look for discrete-harmonic functions on T_α (with respect to the natural Laplacian, which is the same as the generator of the simple random walk on T_α). The condition of balanced embedding is exactly equivalent to saying that the identity map is harmonic on T_α; it is a linear condition, so it does not depend on the value of α.

The main difficulty when looking at discrete holomorphic maps is that the product of two such maps is not holomorphic in general. But we are interested in scaling limits, so maybe imposing that such a product is in fact "almost discrete holomorphic" (in the sense that it satisfies the Cauchy-Riemann equations up to an error term which vanishes in the scaling limit) would be sufficient.

Whether the previous paragraph makes sense or not — let us investigate whether the map $\zeta : z \mapsto z^2$ is discrete-harmonic. For every $z \in T_\alpha$, we can write

$$\Delta\zeta(z) = \frac{1}{3}\sum_{z'\sim z}(z'^2 - z^2) = \frac{1}{3}\sum_{e\in E_z(T)}(z+e_\alpha)^2 - z^2 = \frac{1}{3}\sum_{e\in E_z(T)} e_\alpha^2$$

(the term in $\sum z e_\alpha$ vanishes because the embedding is balanced). So, if ζ is discrete-harmonic, summing the above relation over $z \in T$ gives the very same condition $\sum e_\alpha^2 = 0$ as before; in other words, α_T^{RW} is the embedding for which $z \mapsto z^2$ is *discrete-harmonic on average*.

As a last remark, let us investigate how strong the condition of exact harmonicity of ζ is; so assume that α is chosen in such a way that $\Delta\zeta$ is identically 0. Let e be any oriented edge of T; let $e' := \tau.e$ and $e'' := \tau^2.e$ be the two other edges sharing the same source as e. The balance condition on the embedding plus harmonicity of ζ imply the following system:

$$\left\{ \begin{array}{rcl} e_\alpha + e'_\alpha + e''_\alpha & = & 0 \\ e_\alpha^2 + (e'_\alpha)^2 + (e''_\alpha)^2 & = & 0 \end{array} \right. \tag{2.2}$$

Up to rotation and scaling, one can always assume that $e_\alpha = 1$, so the system reduces to $e'_\alpha + e''_\alpha = -1$ and $(e'_\alpha)^2 + (e''_\alpha)^2 = -1$. Squaring the first of these two relations and substracting the second, one obtains $e'_\alpha e''_\alpha = 1$, so e'_α and e''_α are the two solutions of the equation

$$X^2 + X + 1 = 0$$

which implies that $\{e'_\alpha, e''_\alpha\} = \{e^{\pm 2\pi i/3}\}$. To sum it up:

Proposition 9. *The only* 3*-regular graph on which the map $\zeta : z \mapsto z^2$ is discrete-harmonic is the honeycomb lattice, embedded in such a way that its faces are regular hexagons.*

So, imposing ζ to be harmonic not only determines the embedding, it also restricts T to essentially one graph; but in terms of scaling limits, the condition that ζ is harmonic on the average makes as much sense as the exact condition.

2.4. Embedding using circle packings

There is another way to specify essentially unique embeddings of triangulations, which is very strongly related to conformal geometry, using the theory of circle packings. It is a fascinating subject in itself and a detailed treatment would be outside of the purpose of these notes, so the interested reader is advised to consult the book of Stephenson [18] and the references therein for the proofs of the claims in this subsection and much more.

We first give a version of a theorem of Köbe, Andreev and Thurston, specialized to our case. It is a statement about triangulations, which is why we actually apply it to T^* instead of directly to T. Notice that we *do not* assume T to be already embedded into the torus $\mathbb{T}^2$.

Theorem 10 (Discrete uniformization theorem [18, p. 51]**).** *Let T^* be a finite triangulation of the torus, and let $\hat{T}^*$ be its universal cover. There exists a locally finite family $(\mathcal{C}_v)_{v \in V(\hat{T}^*)}$ of disks of positive radii and disjoint interiors, satisfying the following compatibility condition: $\mathcal{C}_v$ and $\mathcal{C}_{v'}$ are tangent if, and only if, v and v' are neighbors in $\hat{T}^*$.*

Such a family is called a circle packing *associated to the graph $\hat{T}^*$. It is* essentially unique*, in the following sense: If $(\mathcal{C}'_v)$ is another circle packing associated to $\hat{T}^*$, then there is a map $\varphi : \mathbb{C} \to \mathbb{C}$, either of the form $z \mapsto az + b$ or of the form $z \mapsto a\bar{z} + b$, such that for every $v \in V(\hat{T}^*)$, $\mathcal{C}'_v = \varphi(\mathcal{C}_v)$.*

Remark 11. The "existence" part of the above theorem remains true in a much broader class of graphs; essentially all that is necessary is bounded degree and recurrence of the simple random walk on it. (One can see that a packing exists by completing the graph into a triangulation.) The "uniqueness" part however fails in general, as is made clear as soon as one tries to construct a circle packing associated to the square lattice ...

A consequence of the uniqueness part of the theorem is the following: Let $\theta : \hat{T}^* \to \hat{T}^*$ be a translation along one of the periods of $\hat{T}^*$, and let $\mathcal{C}'_v := \mathcal{C}_{\theta(v)}$; according to the theorem, let φ be such that $\mathcal{C}'_v = \varphi(\mathcal{C}_v)$ for all v. Up to composition of φ by itself, one can always assume that it is of the form $\varphi(z) = az + b$. By the assumption of local finiteness of the circle packing, one has $|a| = 1$; besides, the orbits of θ are unbounded, so those of φ are too, and in particular it does not have a fixed point, which implies that $a = 1$ and $b \neq 0$. In other words, φ is a translation, *i.e.* the circle packing associated to $\hat{T}^*$ is itself doubly periodic.

As soon as one is given a circle packing associated to a planar graph, it comes with a natural embedding: Every vertex $v \in V(\hat{T}^*)$ will be represented by the center of $\mathcal{C}_v$, and if v' is a neighbor of v, the edge (v, v') will be embedded as a segment — which is the union of a radius of $\mathcal{C}_v$ and a radius of $\mathcal{C}_{v'}$, because those two disks are tangent. One can then specify an embedding of $\hat{T}$ by putting each of its vertex at the center of the disk inscribed in the corresponding triangular face of (the embedding of) $\hat{T}^*$; the collection of all those inscribed disks is in fact a circle packing associated with the graph $\hat{T}$ (see Figure 3).

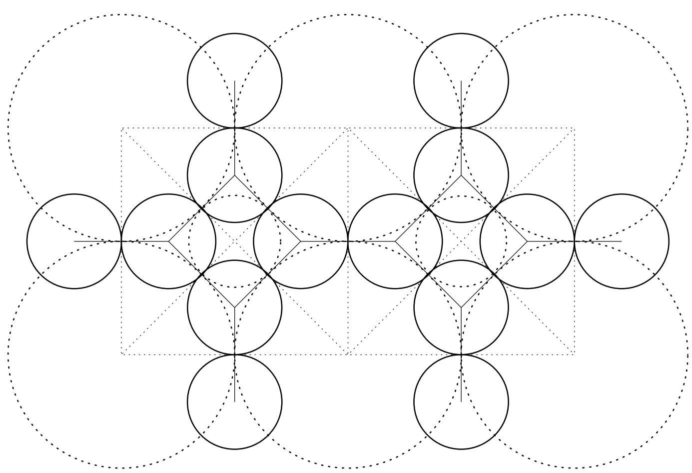

FIGURE 3. The circle packings associated to the graph T_s (solid lines) and its dual (dotted lines).

Of more interest to us is the fact that the embedding is itself doubly periodic, by the previous remarks. Up to rotation, and scaling and maybe complex conjugation, one can assume that the period corresponding to the translation by 1 (resp. i) in $\mathbb{T}^2$ is equal to 1 (resp. $\alpha \in \mathbb{H}$). Once again, the value of the modulus α is uniquely determined; for further reference, we will denote it by α_T^{CP}.

Yet again, as soon as one additional symmetry is present in $\hat{T}$, the value of α_T^{CP} is the same as that obtained using the symmetry; this is again a direct consequence of the essential uniqueness of the circle packing.

2.5. "Exotic" embeddings

Looking closely at Smirnov's proof, one notices that essentially the only place where the specifics of the graph are used is in the proof of "integrability" or exact cancellation; we will come back to this in the next section, let us just mention that the key ingredient in the phenomenon can be seen to be the fact that $\psi(e)$ (as introduced earlier) is identically 0. This is equivalent to saying that all the triangles of the triangular lattice are equilateral.

A way to try and generalize the proof is to demand that all the faces of T_α^* be equilateral triangles. Of course this cannot be done by embedding it in the plane, even locally — the total angle around a vertex would be equal to 2π only if the degree of the vertex is 6. But one can build a 2-dimensional manifold M_T with conic singularities by gluing together equilateral triangles according to the combinatorics of $\hat{T}^*$; since the average degree of a vertex of T^* is equal to 6, the average curvature of the manifold (defined *e.g.* as the limit of the normalized total curvature in large discs) is 0.

The manifold M_T is not flat in general (the only case where it is being the triangular lattice), but it is homeomorphic to the complex plane, and one can hope to see it as a perturbation of it on which some of the standard tools of complex analysis could have counterparts — the optimal being to be able to perform Smirnov's proof within it. This is no easy task, and is probably not doable anyway.

To relate M_T to the topic of this section, one can try to define a module out of it. A good candidate for that is the following: Assume that M_T can be realized as a sub-manifold of $\mathbb{R}^3$ (or in $\mathbb{R}^d$ for $d > 2$ large enough), in such a way that the (combinatorial) translations on $\hat{T}$ act by global translations of the ambient space, thus forming a *periodic sub-manifold*. Then there is a copy of $\mathbb{Z}^2$ acting on it, and the affine plane containing a given point of M_T and spanned by the directions of the two generators of that group is at finite Hausdorff distance from it; in other words, this realization of M_T looks like a bounded perturbation of a Euclidean plane.

One can then look at the orthogonal projections of the vertices of $\hat{T}^*$ (seen as points of M_T) onto that plane; this creates a doubly periodic, locally finite family of points of the Euclidean plane. It is not always possible to form an embedding

of $\hat{T}^*$ in the plane from it (with disjoint edges); but it does define a value of α as above.

Unfortunately, there are cases when this value of α is not well-defined, in the sense that it depends on the choice of M_T; this happens if the (infinite) polyhedron associated to $\hat{T}^*$, with equilateral faces, is *flexible*. The simplest example of this phenomenon is to take T to be two periods of the honeycomb lattice in each direction.

2.6. Comparing different methods of embedding

We now have at least two (forgetting about the last one) ways of giving a conformal structure to a torus equipped with a triangulation — which is but another way of referring to the choice of α. Assuming that critical percolation does have a scaling limit, it leads to a third choice α_T^{Perc} of it.

It would be a natural intuition that all these moduli are the same, and correspond to a notion of *conformal embedding* of a triangulation (or a 3-regular graph) in the plane; and they all have a claim to that name. But this is not true in general: We detail the construction of a counterexample. Start with the graph T_s and its dual T_s^*; and refine one of the "vertical" triangular faces of T_s^* by adding a vertex in the interior of it, connected to its three vertices. In terms of the primal graph, this correspond to replacing one of its vertices by a triangle — see Figure 4. Let T_s' be the graph obtained that way; we will refer to such a splitting as a *refinement*, and to the added vertex as a *new vertex*.

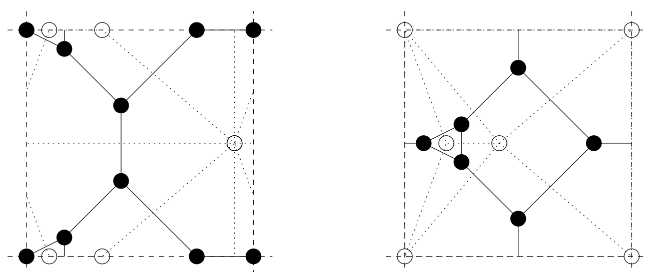

FIGURE 4. Square (but not balanced) embeddings of T_s' (solid) and its dual (dotted); the origin is taken as a point of T_s' on the left, and as a point of the dual on the right, corresponding to the ones chosen for Figures 1 and 2.

In terms of circle packings, this changes essentially nothing; the new vertex of $(T_s')^*$ can be realized as a new disc without modifying the rest of the configuration (cf. Figure 5). In terms of random walks, however, adding edges will modify the covariance matrix in the central limit theorem. The computation can be done

easily, as explained above, and one gets the following values:

$$\alpha^{\mathrm{CP}}_{T'_s} = i = \alpha^{\mathrm{CP}}_{T_s} \; ; \qquad \alpha^{\mathrm{RW}}_{T'_s} = i\sqrt{\frac{6}{7}} \neq i = \alpha^{\mathrm{RW}}_{T}.$$

In this particular case, the value of $\alpha^{\mathrm{RW}}_{(T'_s)^*}$ is also $i\sqrt{6/7}$.

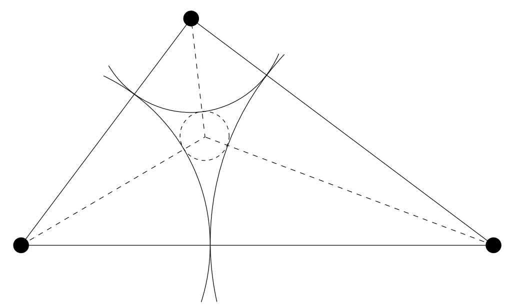

FIGURE 5. Splitting a face of a triangulation into 3 triangles, and the corresponding modification of its circle packing (added features represented by dashed lines)

So, α^{RW} and α^{CP} are different in general. Is α^{Perc} (provided it exists) one of them? An easy fact to notice is the following: Let T^* be a triangulation of the torus and let $(T')^*$ be obtained from it by splitting a triangle into 3 as in the construction of T'_s. Then, consider two realizations of site-percolation at $p_c = 1/2$ on both universal covers, coupled in such a way that the common vertices are in the same state for both models. In other words, start with a realization of percolation on $\hat{T}^*$ and without changing site states, refine a periodic family of triangles of it into 3, choosing the state of each new vertex independently of the others and of the configuration on $\hat{T}^*$.

If there is a chain of open vertices in $\hat{T}^*$, this chain is also a chain of open vertices in the refined graph — because all the edges are preserved in the refinement. Conversely, starting from a chain of open vertices in the refinement and removing each occurrence of a new vertex on it, one obtains a chain of open vertices in $\hat{T}^*$; the reason for that being that the triangle is a complete graph. Another way of stating the same fact is to say that opening (resp. closing) one of the new vertices cannot join two previously disjoint open clusters (resp. split a cluster into two disjoint components); they cannot be *pivotal* for a crossing event.

Hence, the probability that a large conformal rectangle is crossed is the same in both cases (at least if the choice of the discrete approximation of its boundary is the same for both graphs, which in particular implies that it contains no new vertex), and so is f_α for every choice of α (still assuming that it exists, of course).

If one is conformally invariant in the scaling limit, the other also has to be. In short,

$$\alpha_T^{\mathrm{Perc}} = \alpha_{T'}^{\mathrm{Perc}}.$$

Looking at circle packings instead of percolation, we get the same identity (as was mentioned in the particular case of T_s), with a very similar proof: Adding a vertex does not change anything to the rest of the picture, and we readily obtain

$$\alpha_T^{\mathrm{CP}} = \alpha_{T'}^{\mathrm{CP}}.$$

This leads us to the following hope, which we state as a conjecture even though it is much closer to being wishful thinking:

Conjecture 12. *Let T^* be a triangulation of the torus. Then, the critical parameter for site-percolation on its universal cover $\hat{T}^*$ is equal to $1/2$, and for every $\alpha \in \mathbb{H}$, critical site-percolation on $\hat{T}^*_\alpha$ has a scaling limit. The value of the modulus α for which the model is conformally invariant in the scaling limit is that obtained from the circle packing associated to $\hat{T}^*$:*

$$\alpha_T^{\mathrm{Perc}} = \alpha_T^{\mathrm{CP}}.$$

3. Critical percolation on the triangular lattice

For reference, and as a way of introducing our general strategy, we give in this section a very shortened version of Smirnov's proof of the existence and conformal invariance of the scaling limit for critical site-percolation on the triangular lattice T_h. The interested reader is advised to consult our previous note [3] for an "extended shortening", or Smirnov's article [16] for the original proof; see the book of Bollobás and Riordan [5] for a more detailed treatment. Up to cosmetic changes, we follow the notation of [3].

Remark 13. Up to the last paragraph of the section, we are not assuming that the lattice we are working with is the honeycomb lattice; our only assumption is that we have an *a priori* bound for crossing probabilities of large rectangles which depends on their aspect ratio but not on their size (we "assume Russo-Seymour-Welsh conditions"). It is not actually clear how general those are; all the standard proofs require at least some symmetry in the lattice in addition to periodicity, but it is a natural conjecture that periodicity is enough.

Here and in the remainder of this paper, $\tau := e^{2\pi i/3}$ will be the third root of unity with positive imaginary part. Let T be a finite graph of genus 1, T_α an embedding of modulus α of T in the complex plane; let $V(T_\alpha)$ (resp. $E(T_\alpha)$) be the set of vertices (resp. *oriented* edges) of T_α. Each vertex $z \in V(T_\alpha)$ has three neighbours; let $E_z(T_\alpha)$ be the set of the three oriented edges in $E(T_\alpha)$ having their source at z. That set can be cyclically ordered counterclockwise; if $e \in E_z(T_\alpha)$ is one of the three edges starting at z, we will denote by $\tau.e$ (resp. $\tau^2.e$) the next (resp. second to next) edge in the ordering.

Remark 14. In the particular case of the honeycomb lattice, seeing each edge as a complex number (being the difference between its target and its source), the notation $\tau.e$ corresponds to complex multiplication by $e^{2\pi i/3}$ — in other words, $\tau.e = \tau e$ as a product of complex numbers. That is of course not the case in general, but we keep the formal notation for clarity. In what follows, whenever an algebraic expression involves the product of a complex number by an edge of T_α or T_α^*, as above the edge will be understood as the difference, as a complex number, between its target and its source; we will never use formal linear combinations of edges. The notation $\tau.e$ (with a dot) will be reserved for the "topological" rotation within $E_z(T_\alpha)$.

Let again Ω be a smooth Jordan domain in the complex plane, and let A, B, C and D be three points on its boundary, in that order when following $\partial\Omega$ counterclockwise. Let Ω_δ be the largest connected component of $\Omega \cap \delta T_\alpha$, and let A_δ (resp. B_δ, C_δ, D_δ) be the point of Ω_δ that is closest to A (resp. B, C, D). The main result in Smirnov's paper ([16]) is the following:

Theorem 15 (Smirnov). *In the case where T_α is the honeycomb lattice, embedded so as to make its faces regular hexagons (*i.e.*, when $\alpha = i\sqrt{3}/3$), critical site-percolation has a conformally invariant scaling limit. If Ω is an equilateral triangle with vertices A, B and C, then*

$$f_\alpha(\Omega, A, B, C, D) = \frac{|CD|}{|CA|}.$$

Knowing this particular family of values of f_α is enough, together with conformal invariance, to compute it for any conformal rectangle. The formula obtained for a rectangle is known as *Cardy's formula.*

To each edge $e \in E(T_\alpha)$ corresponds its dual oriented edge $e^* \in E(T_\alpha^*)$, oriented in such a way that the angle (e, e^*) is in $(0, \pi)$. If $-e$ denotes the edge with the same endpoints as e but the reverse orientation, then we have $e^{**} = -e$. Define

$$\psi(e) := e^* + \tau(\tau.e)^* + \tau^2(\tau^2.e)^*$$

(where as above we interpret the edges e^*, $(\tau.e)^*$ and $(\tau^2.e)^*$ as complex numbers). It is easy to check that $\psi(e) = 0$ if, and only if, the face of T_α^* corresponding to the source of e is an equilateral triangle; so, $\psi(e)$ can be seen as a measure of the local deviation between T_α and the honeycomb lattice. An identity which will be useful later is the following:

$$\forall z \in V(T_\alpha), \qquad \sum_{e \in E_z(T_\alpha)} \psi(e) = 0. \tag{3.1}$$

For every $z \in \Omega_\delta$, let $E_{A,\delta}(z)$ be the event that there is a simple path of open vertices of Ω_δ^*, joining two points of the boundary of the domain, which separates

z and A from B and C; let $H_A := P[E_{A,\delta}(z)]$. Define similar events for points B and C by a circular permutation of the letters, and let

$$S_\delta(z) := H_{A,\delta}(z) + H_{B,\delta}(z) + H_{C,\delta}(z),$$
$$H_\delta(z) := H_{A,\delta}(z) + \tau H_{B,\delta}(z) + \tau^2 H_{C,\delta}(z).$$

It is a direct consequence of Russo-Seymour-Welsh estimates that these functions are all Hölder with some universal positive exponent, with a norm which does not depend on δ, so by Ascoli's theorem they form a relatively compact family, and as $\delta \to 0$ they have subsequential limits which are Hölder maps from Ω to $\mathbb{C}$; all that is needed is prove that only one such limit is possible.

The key argument is to show that if h (resp. s) is any subsequential limit of (H_δ) (resp. (S_δ)) as $\delta \to 0$, then h and s are holomorphic; indeed, assume for a moment that they are. Since s is also real-valued, it has to be constant, and its value is 1 by boundary conditions (*e.g.* at point A). On the other hand, along the boundary arc $(A_\delta B_\delta)$ of $\partial\Omega_\delta$, $H_{C,\delta}$ is identically 0, so the image of the arc (AB) by h is contained in the segment $[1,\tau]$ of $\mathbb{C}$; and similar statements hold *mutatis mutandis* for the arcs (BC) and (CA). By basic index theory, this implies that h is the unique conformal map sending Ω to the (equilateral) triangle of vertices 1, τ and τ^2, and that is enough to characterize it and to finish the proof of Theorem 15.

So, the crux of the matter, as expected, is to prove that the map h has to be holomorphic. The most convenient way to do that is to use Morera's theorem, which states that h is indeed holomorphic on Ω if, and only if, its integral along any closed, smooth curve contained in Ω is equal to 0.

Let γ be such a curve, and let $\gamma_\delta = (z_0, z_1, \dots, z_{L_\delta} = z_0)$ be a closed chain of vertices of Ω_δ which approximates it within Hausdorff distance δ and has $\mathcal{O}(\delta^{-1})$ points. Because the functions H_δ are uniformly Hölder, it follows that

$$\oint_{\gamma_\delta} H_\delta(z)\mathrm{d}z := \sum_{k=0}^{L_\delta - 1} H_\delta(z_k)(z_{k+1} - z_k) \to \oint_\gamma h(z)\mathrm{d}z.$$

We want to prove that, for a suitable choice of α, the discrete integral on the left-hand side of that equation vanishes in the scaling limit.

If $e = (z, z')$ is an oriented edge of Ω_δ, define $P_{A,\delta}(e) := P[E_{A,\delta}(z') \setminus E_{A,\delta}(z)]$; define $P_{B,\delta}$ and $P_{C,\delta}$ similarly. A very clever remark due to Smirnov, which is actually the only place in his proof where specifics of the model (as opposed to the lattice) are used, is that one can use color-swapping arguments to prove that, for every oriented edge,

$$P_{A,\delta}(e) = P_{B,\delta}(\tau.e) = P_{C,\delta}(\tau^2.e). \tag{3.2}$$

On the other hand, since differences of values of H_δ between points of Ω_δ can be computed in terms of these functions $P_{\cdot,\delta}$, the discrete integral above can be

rewritten using them: Letting $E(\gamma_\delta)$ be the set of edges contained in the domain surrounded by γ_δ and using (3.2), one gets

$$\oint_{\gamma_\delta} H_\delta(z)\mathrm{d}z = \sum_{e\in E(\gamma_\delta)} \psi(e)P_{A,\delta}(e) + o(1). \tag{3.3}$$

A similar computation, together with the fact that $e^*+(\tau.e)^*+(\tau^2.e)^*$ is identically equal to 0, leads to

$$\oint_{\gamma_\delta} S_\delta(z)\mathrm{d}z = o(1). \tag{3.4}$$

We again refer the reader to [3] for the details of this construction.

Notice that it already implies that s is holomorphic, hence constant equal to 1, independently of the value of α; so, whether h is holomorphic or not, it will send $\bar\Omega$ to the triangle of vertices 1, τ and τ^2 anyway. In the case of the triangular lattice embedded in the usual way, $\psi(e)$ is also identically equal to 0, as was mentioned above, so h is itself holomorphic, and the proof is complete.

The remainder of these notes is devoted to some ideas about how to extend the general framework of the proof to more general cases; it is not clear how close one is to a proof, but it is likely that at least one fundamentally new idea will be required. However, we do believe that the overall strategy which we will now describe is the right angle of attack of the problem. Do not expect to find any formal proof in what follows, though.

4. Other triangulations

4.1. Using local shifts

The first natural idea when trying to generalize the construction of Smirnov is to try an apply it to more general periodic triangulations of the plane. Indeed, in all that precedes, up to and including Equation (3.3), nothing is specific to the regular triangular lattice, only Russo-Seymour-Welsh conditions (and their corollary that $p_c = 1/2$) are needed. It is only at the very last step, noticing that ψ was identically equal to 0, that the precise geometry was needed.

The key fact that makes hope possible is the following (and it is actually similar to one of the points we made earlier): In the expression of the discrete integral as a sum over interior edges, each term is the product of two contributions:

- $\psi(e)$ which depends on the geometry of the embedding, and through that on the value of α;
- $P_{A,\delta}(e)$ which is only a function of the combinatorics of Ω_δ.

Even though Ω_δ as a graph does depend on the choice of α, one can make the following remark: Applying the transformation φ_β (for some $\beta \in \mathbb{H}$) to both the domain Ω and the lattice δT_α does not change Ω_α as a graph. In particular it does not change the value of $P_{A,\delta}(e)$.

One can then see the whole sum as a function β, say $S_{\Omega,\delta}(\beta)$. Because $\varphi_\beta(z)$ is a real-affine function of β, so is $S_{\Omega,\delta}$; one can then try to solve the equation $S_{\Omega,\delta}(\beta) = 0$ in β. Using the corresponding φ_β, one gets a joint choice of a domain, a lattice modulus and mesh, and a curve γ making the discrete contour integral vanish.

It the modulus thus obtained actually did not depend on Ω, δ or γ, we would be done — call it α_T^{Perc} and there is only bookkeeping left to do. However we do not even know whether it has a limit as $\delta \downarrow 0$ ... An alternative is as follows. Because the lattice is periodic, it makes sense to first look at the sum $\sum \psi(e) P_{A,\delta}(e)$ over one fundamental domain. If that is small, then over the copy of the fundamental domain immediately to the right of the previous one, the terms $\psi(e)$ are exactly the same, and one is lead to compare $P_{A,\delta}$ for two neighboring pre-images of a given edge of T.

So, let e be an edge of Ω_δ, and let $e + \delta$ be its image by a translation of one period to the right. Making the dependency on the shape of the domain explicit in the notation, one can replace the translation of e by a translation of the domain itself and the boundary points in the opposite direction, to obtain

$$P^{\Omega}_{A,\delta}(e + \delta) = P^{\Omega-\delta}_{A,\delta}(e). \tag{4.1}$$

To estimate the difference between this term and the corresponding one in Ω, one can consider coupling two realizations of percolation, one on Ω_δ and the other in $\Omega_\delta - \delta$, so that they coincide on the intersection between the two.

The event corresponding to $P^{\Omega}_{A,\delta}(e)$ is that there is an open simple path separating the target of e and A from B, and C, and that no open simple path separates the source of e and A from B and C; this is equivalent to the existence of 3 disjoint paths from the 3 vertices of the face at the source of e to the 3 "sides" of the conformal triangle (Ω, A, B, C), two of them being formed of open vertices and the third being formed of closed vertices — cf. Figure 6. For this to happen in Ω but not in $\Omega - \delta$, one of these arms needs to go up to $\partial\Omega$ but not to $\partial(\Omega - \delta)$, and the only way for this to be realized is for a path of the opposite color to prevent it; this can be done in finitely many ways, Figure 6 being one of them; $P^{\Omega}_{A,\delta} - P^{\Omega-\delta}_{A,\delta}$ can then be written as the linear combination of the probabilities of finitely many terms of that form — half of these actually corresponding to the reversed situation, where arms go up to $\partial(\Omega - \delta)$ but not up to $\partial\Omega$.

In the case corresponding to Figure 6, and all the similar ones, one sees that 3 arms connect the source of e to the boundary of $\Omega \cap (\Omega - \delta)$, and on at least one point of that boundaries there have to be 3 disjoint arms of diameter of order 1. There are $\mathcal{O}(\delta^{-1})$ points on the boundary, and the probability that 3 such arms exist from one of them is known — at least in the case of a polygon, which is enough for our purposes — to behave like δ^2, see *e.g.* [17].

Another possible reason for the non-existence of 3 arms from the source of e to the correct portions of the boundary of $\Omega - \delta$ (say) is that one of the corresponding arms in Ω actually lands very close to either A, B or C: Preventing

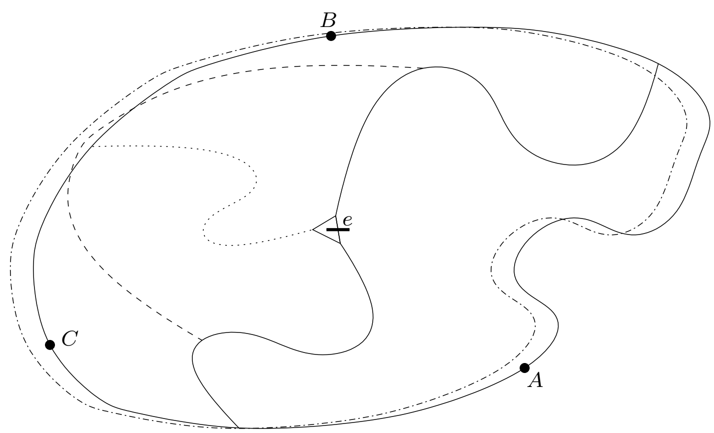

FIGURE 6. A typical case contributing to $P^{\Omega}_{A,\delta}(e) - P^{\Omega-\delta}_{A,\delta}(e)$. The original domain boundary is represented by a solid line, that of the shifted domain bu a dashed-and-dotted line; open (resp. closed) arms from the source of e are represented as solid (resp. dotted) lines, and the additional open path preventing the closed arm from connecting to the boundary of $\Omega_\delta - \delta$ is represented as a dashed curve.

it from touching the relevant part of $\partial(\Omega - \delta)$ requires only *one* additional arm from a δ-neighborhood of that vertex — *i.e.*, a total of 2 arms of diameter of order 1. The probability for that (see [17] also), still in the case when Ω is a polygon with none of A, B or C as a vertex, behaves like δ. Fortunately, there are only 3 corners on a conformal triangle, so the contribution of these cases is of the same order as previously

Putting everything together, one gets an estimate of the form

$$P^{\Omega-\delta}_{A,\delta}(e) = P^{\Omega}_{A,\delta}(e)\left[1 + \mathcal{O}(\delta)\right]. \tag{4.2}$$

Coming back to our current goal, let $\mathcal{E}$ be the set of oriented edges in a given period of Ω_δ, and let $\mathcal{E} + \delta$ be its image by the translation of vector δ. Then,

$$\begin{aligned}\sum_{e\in\mathcal{E}+\delta} \psi(e) P^{\Omega}_{A,\delta}(e) &= \sum_{e\in\mathcal{E}} \psi(e) P^{\Omega}_{A,\delta}(e)\left[1 + \mathcal{O}(\delta)\right] \\ &= \sum_{e\in\mathcal{E}} \psi(e) P^{\Omega}_{A,\delta}(e) + \mathcal{O}(\delta^{2+\eta})\end{aligned}$$

with $\eta > 0$; the existence of such an η is ensured by Russo-Seymour-Welsh type arguments again, which ensure that, uniformly in e and δ, for every edge e, $P^{\Omega}_{A,\delta}(e) = \mathcal{O}(\delta^\eta)$.

Now, if that is the way the proof starts, what needs to be done is quite clear:

- Fix a period $\mathcal{E}$ of the graph,
- Choose α so that the previous sum, over this period, of $\psi(e)P^{\Omega}_{A,\delta}(e)$ is equal to 0,
- Use the above estimate to give an upper bound for the same sum on neighboring periods;
- Try to somehow propagate the estimate up to the boundary.

The last part of the plan is the one that does not work directly, because one needs of the order of δ^{-1} steps to go from $\mathcal{E}$ to $\partial\Omega$, and the previous bound is not small enough to achieve that; one would need a term of the order of $\mathcal{O}(\delta^{3+\eta})$. It is however quite possible that a more careful decomposition of the events would lead to additional cancellation, though we were not able to perform it.

4.2. Using incipient infinite clusters

Another idea which might have a better chance of working out is based on the idea of incipient infinite clusters. We are trying to ensure that $\sum \psi(e)P_{A,\delta}(e)$ is equal to $o(\delta^2)$ over a period for a suitable choice of α; but for it to be exactly equal to 0 depends only on the ratios $P_A(e)/P_A(e')$ within the period considered, and not on their individual values. One can then let δ go to 0, or equivalently let Ω increase to cover the whole space, and look at this ratio.

Proposition 16. *There is a map $\pi : E(\hat{T}) \to (0,+\infty)$ such that the following happens. Let e, e' be two edges of T_α, which we identify with $\hat{T}$ for easier notation, and let $\delta = 1$. Then, as Ω increases to cover the whole plane,*

$$\frac{P^{\Omega}_{A,1}(e)}{P^{\Omega}_{A,1}(e')} \to \frac{\pi(e)}{\pi(e')},$$

uniformly in the choices of A, B and C on $\partial\Omega$. The map π is periodic and does not depend on the choice of α.

Proof. The argument is very similar to Kesten's proof of existence of the incipient infinite cluster (see [11]); it is based on Russo-Seymour-Welsh estimates. It will appear in an upcoming paper [2]. Notice that there is no requirement for A, B and C to remain separated from each other; this is similar to the fact that the incipient infinite cluster is also the limit, as $n \to \infty$, of critical percolation conditioned to the event that the origin is connected to the point $(n, 0)$ — which in turn is again a consequence of Russo-Seymour-Welsh theory. The speed of convergence is certainly different with and without such restrictions on the positions of A, B and C, though. □

Seeing this Proposition, one is tempted to define α by solving the equation

$$\sum_{e\in\mathcal{E}} \psi(e)\pi(e) = 0, \tag{4.3}$$

where again the sum is taken over one period of the lattice. Indeed, all that remains in the sum, over the same period of the lattice, of $\psi(e)P_A(e)$ is composed of terms

of a smaller order. However, because the limit taken to define π is uniform in the choices of A, B and C, in particular it is invariant by re-labelling of the corners of the conformal triangle; equivalently, taking P_B instead of P_A leads to the same limit. Combining this remark with Equation (3.2), one gets the following identities:

$$\forall e \in E(\hat{T}), \quad \pi(e) = \pi(\tau.e) = \pi(\tau^2.e). \tag{4.4}$$

In other words, $\pi(e)$ only depends on the source of e. For every edge $e = (z, z')$, let $\pi(z) := \pi(e)$: If $\mathcal{V}$ is a period of $V(\hat{T})$, one has

$$\sum_{e \in \mathcal{E}} \psi(e)\pi(e) = \sum_{z \in \mathcal{V}} \pi(z) \sum_{z' \sim z} \psi((z, z')) = 0$$

by using the remark in Equation (3.1). So, the equation (4.3) is actually always true, and does not help in finding the value of α ...

This is actually good news, because it is the sign of emerging cancellations in the scaling limit, which were not at first apparent; that means that the relevant terms in (3.3) are actually smaller than they look at first sight, which in turn means that making the leading term equal to 0 by the correct choice of α leads to even smaller terms.

Whether the overall strategy can be made to work actually depends on the speed of convergence in the statement of Proposition 16. In the case of the triangular lattice, one can actually use SLE to give an explicit expansion of the ratio $P_A(e)/P_A(e')$ as Ω increases, at least in some cases; this is the subject of an upcoming paper [2].

5. Other lattices

5.1. Mixed percolation

We conclude the speculative part of these notes by some considerations about bond-percolation on the planar square lattice. The combinatorial construction we perform here does apply to more general cases, but the probabilistic arguments which follow do not, so we restrict ourselves to the case of $\mathbb{Z}^2$.

The general idea it to map the problem of bond-percolation on $\mathbb{Z}^2$ to one of site-percolation on a suitable triangulation of the plane. Then, if the arguments in the previous section can be made to work, one could potentially prove the existence and conformal invariance of a scaling limit of critical percolation on the square lattice.

The key remark was already present in the book of Kesten [10]: For any bond-percolation model on a graph, one can construct the so-called *covering graph* on which it corresponds to site-percolation. More specifically, let G_1 be a connected graph with bounded degree; as usual, let $E(G_1)$ be the set of its edges and $V(G_1)$ be the set of its vertices. We construct a graph G_2 as follows: The set $V(G_2)$ of its vertices is chosen to be $E(G_1)$, and we put an edge between two vertices of G_2

if, and only if, the corresponding edges of G_2 share an endpoint. Notice that even if G_1 is assumed to be planar, G_2 does not have to be — see Figure 7 for the case of $\mathbb{Z}^2$.

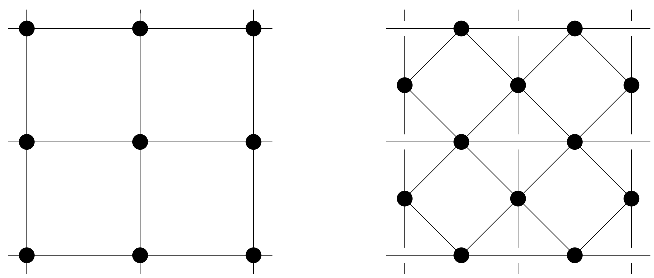

FIGURE 7. The square lattice $\mathbb{Z}^2$ and its covering graph

The graph thus obtained from the square lattice is isomorphic to a copy of the square lattice where every second face, in a checkerboard disposition, is completed into a complete graph with 4 vertices. The next remark is the following: in terms of site-percolation, a complete graph with 4 vertices behaves the same way as a square with an additional vertex at the center, which is open with probability 1 — with the same meaning as when we looked at refinement of triangles in triangulations, *i.e.* taking a chain of open vertices in the partially centered square lattice and removing from it the vertices which are face centers leads to a chain of open vertices in the covering graph of $\mathbb{Z}^2$.

So, let again G_s be the centered square lattice, as was introduced above, and let $q \in [0,1]$; split the vertices of G_s into three classes, to defined a non-homogeneous site-percolation model, as follows. Each vertex is either open or closed, independently of the others, and:

- The sites of $\mathbb{Z}^2$ are open with probability $p = 1/2$; we will call them *vertices of type I*, or p-sites for short, and denote by V_1 the set of such vertices;
- The vertices of coordinates $(k+1/2, l+1/2)$ with $k+l$ even are open with probability q; we will call them *vertices of type II*, or q-sites for short, and denote by V_2 the set of such vertices;
- The vertices of coordinates $(k+1/2, l+1/2)$ with $k+l$ odd are open with probability $1-q$; we will call them *vertices of type III*, or $(1-q)$-sites for short, and denote by V_3 the set of such vertices.

We will refer to that model as *mixed percolation* with parameters $p = 1/2$ and q, and denote by $P_{1/2,q}$ the associated probability measure. Two cases are of particular interest:

- If $q = 1/2$, the model is exactly critical site-percolation on the centered square lattice G_s;

- If $q = 0$ or $q = 1$ (the situation is the same in both cases up to a translation), from the previous remarks mixed percolation then corresponds to critical bond-percolation on the square lattice.

Besides, all the models obtained for $p = 1/2$ are critical and satisfy Russo-Seymour-Welsh estimates.

5.2. Model interpolation

We are now equipped to perform an interpolation between the models at $q = 0$ and $q = 1/2$. Let (Ω, A, B, C, D) be a simply connected subset of $\mathbb{Z}^2$ equipped with 4 boundary points — say, a rectangle; let $U = U_{\Omega,A,B,C,D}$ be the event, under mixed percolation with parameters $p = 1/2$ and q, that there is a chain of open vertices of Ω joining the boundary arcs (AB) and (CD). To estimate the difference between the probabilities of U for the two models we are most interested in, simply write

$$P_{1/2,1/2}[U] - P_{1/2,0}[U] = \int_0^{1/2} \frac{\partial}{\partial q} P_{1/2,q}[U] \, \mathrm{d}q. \tag{5.1}$$

If percolation is indeed universal, then one would expect cancellation to occur, hopefully for each value of q; the optimal statement being of the form

$$\lim_{\Omega \uparrow \mathbb{Z}^2} \sup_{A,B,C,D \in \partial\Omega} \sup_{q \in (0,1)} \frac{\partial}{\partial q} P_{1/2,q}[U] = 0. \tag{5.2}$$

The main ingredient in the estimation of the derivative in q is, as one might expect, a slight generalization of Russo's formula; to state it, we need a definition:

Definition 17. Consider mixed percolation on G_s, and let E be a cylindrical increasing event for it (*i.e.*, an event which depends on the state of finitely many vertices). Given a realization ω of the model, we say that a vertex v is *pivotal for the event* E if E is realized for the configuration ω^v where v is made open, and not realized for the configuration ω_v where v is made closed. We will denote by $\mathrm{Piv}(E)$ the (random) set of pivotal vertices for E.

Proposition 18. *With the above notation, one has*

$$\frac{\partial}{\partial q} P_{1/2,q}[U] = E_{1/2,q}\left[|\mathrm{Piv}(U) \cap \Omega \cap V_2| - |\mathrm{Piv}(U) \cap \Omega \cap V_3|\right].$$

Proof. The argument is the same as in the proof of the usual formula (in the case of homogeneous percolation); we refer the reader to the book of Grimmett [8]. □

As was the case in the previous section, one can relate the event that a given site is pivotal to the presence of disjoint arms in the realization of the model, with appropriate color. More precisely, a q-site (say) $v \in \Omega$ is pivotal if, and only if, the following happens: v is at the center of a face of $\mathbb{Z}^2$; two opposite vertices of that face are connected respectively to the boundary arcs (AB) and (CD) by disjoint chains of open vertices; the other two vertices of the face are connected respectively to the boundary arcs (BC) and (AD) by disjoint chains of closed vertices; and none

of the chains involved contains the vertex v. To state the previous description more quickly, there is a 4-arm configuration with alternating colors at vertex v, and the endpoints of the arms are appropriately located on $\partial\Omega$ — see Figure 8.

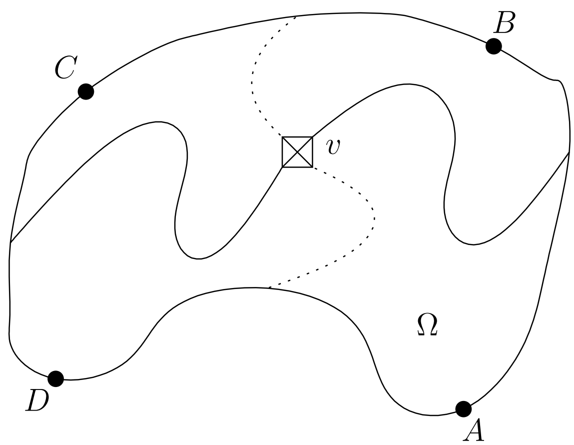

FIGURE 8. A four-arm configuration at vertex v making it pivotal for the event $U(\Omega, A, B, C, D)$.

The main feature of mixed percolation in the case of the centered square lattice is the following: Starting from a configuration sampled according to $P_{1/2,q}$ and shifting the state of all vertices by one lattice mesh to the right, or equivalently flipping the state of all vertices, or rotating the whole configuration by an angle of $\pi/2$ around a site of type I, one gets a configuration sampled according to $P_{1/2,1-q}$; on the other hand, rotating the picture by $\pi/2$ around a vertex of type II or III leaves the measure invariant.

Notice that the existence of 4 arms of alternating colors from a given vertex v is invariant by color-swapping; the configuration in Figure 8 is not though, because the arms obtained after the color change connect the neighbors of v to the wrong parts of the boundary. Nevertheless, one can try to apply the same reasoning as in the previous section, as follows: Let v' be the vertex that is one lattice step to the right of v. If v is a vertex of type II, then v' is a vertex of type III, and up to boundary terms, one can pair all the q-sites in Ω to corresponding $(1-q)$-sites.

To estimate the right-hand term in the statement of Proposition 18, let

$$\Delta(v) := P[v \in \mathrm{Piv}(U)] - P[v' \in \mathrm{Piv}(U)].$$

Our goal will be achieved if one is able to show that $\Delta(v) = o(|\Omega|^{-1})$; or equivalently, if Ω_δ is obtained from a fixed continuous domain by discretization with mesh δ, if one has

$$\Delta(v) = o(\delta^2).$$

In the case of critical site-percolation on the triangular lattice, arguments using SLE processes give an estimate to the probability that a vertex is pivotal, and from universality conjectures it is natural to expect that they extend to the case of mixed percolation on T_s. They involve the 4-arm exponent of percolation, and would read (still in the case of a fixed domain discretized at mesh δ) as

$$P[v \in \mathrm{Piv}(U)] \approx \delta^{5/4}.$$

So, shifting the domain instead of the point as we did in the last section, one would expect an estimate on $\Delta(v)$ of the order

$$\Delta(v) \approx \delta^{9/4}$$

(where the addition of 1 in the exponent corresponds to the presence of a 3-arm configuration at some point on the boundary on either the original domain or its image by the shift). Since $9/4 > 2$, that would be enough to conclude.

However, this approach does not work directly, because of the previous remark that the shift by one lattice step does change the measure, replacing q by $1-q$. If one is interested in the mere existence of the 4 arms around a vertex, combining the shift with color-flipping is enough to cancel the effect; but the estimate one obtains that way is of the form

$$P[v \in \mathrm{Piv}(U_{\Omega,A,B,C,D})] - P[v' \in \mathrm{Piv}(U_{\Omega,B,C,D,A})] \approx \delta P[v \in \mathrm{Piv}(U_{\Omega,A,B,C,D})] \tag{5.3}$$

(and Russo-Seymour-Welsh estimates are actually enough to obtain a formal proof of this estimate).

So, once again, what is missing is a way to estimate how much $P[v \in \mathrm{Piv}(U_{\Omega,A,B,C,D})]$ depends on the location of A, B, C and D along $\partial\Omega$; if the dependency is very weak, then the estimate in Equation (5.3) might actually be of the right order of magnitude. Once again, it is likely that the way to proceed is to use a modified version of the incipient infinite cluster conditioned to have 4 arms of alternating colors from the boundary, and that the order of magnitude of $\Delta(v)$ will be related to the speed of convergence of conditioned percolation to the incipient clusters; but we were not able to conclude the proof that way. It would seem that this part of the argument is easier to formalize than that of the previous section, though, and hopefully a clever reader of these notes will be able to do just that ...

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